Random Artin Groups

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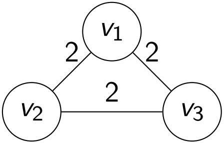
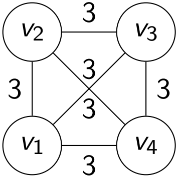
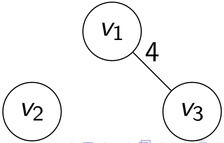
Recall an Artin Group $A(\Gamma)$ can be described in terms of an edge-weighted graph Γ (with weights ≥ 2):

$\mathbb{Z}^3 \leftrightarrow$ 

$\langle v_1, v_2, v_3, v_4 | v_i v_j v_i = v_j v_i v_j \rangle \leftrightarrow$ 

$\langle v_1, v_2, v_3 | v_3 v_1 v_3 v_1 = v_1 v_3 v_1 v_3 \rangle \leftrightarrow$ 

Graphs and Groups

The properties of Artin Groups are often determined by the properties of their defining graphs:

Examples

$$A(\Gamma) \text{ splits as free product } \iff \Gamma \text{ is disjoint union}$$
$$A(\Gamma) \text{ splits as direct product } \iff \Gamma \text{ is join along 2-edges}$$
$$A(\Gamma) \text{ is } \pi_1 \text{ of a 3-fold } \iff \Gamma \text{ is union of trees and triangles of 2-edges}$$

We can use this correspondence to find the probabilities of properties of Artin Groups defined by random graphs.

A Random Graph Model

(Deibel, 2020)

Let $n \in \mathbb{Z}^+$ be given, and let $p_2, p_3, ... \in [0, 1]$ be such that $0 \leq \sum p_i \leq 1$.

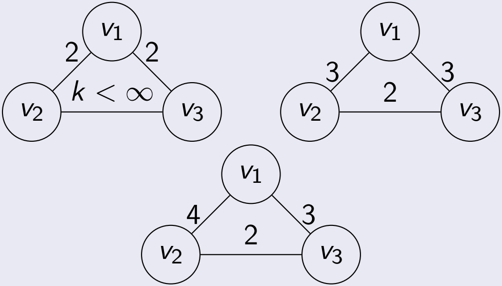
Take the complete unlabeled graph on n vertices, and assign each edge a weight i with probability p_i, or omit it with probability $1 - \sum p_i$.

We often consider $p_2, p_3...$ to be non-constant functions of n, and consider the asymptotic behavior as $n \to \infty$.

Two Dimensional Artin Groups

Theorem (Charney and Davis, 1990's)

An Artin Group has two-dimensional $K(\pi, 1)$ space if and only if it does not contain any of the following subgraphs:



Two Dimensional Artin Groups (cont'd)

Theorem

Let $P_{2d}(n)$ denote the probability that $A(\Gamma)$ has two-dimensional $K(\pi, 1)$ space. If

$$\sum_{k=2,3,\ldots} n^3 p_2(n)^2 p_k(n) \to 0,$$

$$n^3 p_3(n)^2 p_2(n) \to 0 \text{ and } n^3 p_2(n) p_3(n) p_4(n) \to 0,$$

then $P_{2d}(n) \to 1$.

Proof sketch: Each of these three expressions is an upper bound for the expected number of triangles of the above forms (see Deibel (2020)). By the first moment method, the probability that there are no such triangles therefore tends to 1, hence $P_{2d}(n) \to 1$. □

Bibliography

- Deibel, A. (2020). Random coxeter groups. International Journal of Algebra and Computation, 30(06), 1305-1321. doi:10.1142/s0218196720500423